Involutive-like Bases for Rees Ideals of Monomial Curves

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Motivation

- Involutive bases
  - Gröbner bases with additional combinatorial properties
  - Contain elements not needed for minimal Gröbner basis
  - Often highly redundant for binomial ideals

- Rees ideals of monomial curves
  - Binomial ideals with rich combinatorial structure
  - Encode syzygies of all powers of parametrization ideal
  - Free resolutions difficult to describe in most cases

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Aim

Find suitable term order and sparse linear change of coordinates to obtain Pommaret-like bases for large classes of Rees ideals.

Acknowledgements

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- Amir Hashemi and Werner Seiler (involutive and involutive-like bases),
- Rodrigo Iglesias, Eduardo Sáenz de Cabezón, and Werner Seiler (Rees ideals of monomial curves).

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Notation

- $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$: polynomial ring
- $\mathcal{T}$: set of terms $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$.
- $\mathrm{lt}(f)$: largest term of polynomial $0 \neq f \in \mathcal{R}$ w.r.t. fixed term ordering

Involutive Divisions and Bases

Involutive division L: Restricted division relation for terms.

- Given set $H \subset \mathcal{T}$ and $\mathbf{x}^\mu \in H$, assigns multiplicative variables $L(\mathbf{x}^\mu, H) \subseteq \{x_1, \ldots, x_n\}$.
- The involutive cones $\mathcal{C}_L(\mathbf{x}^\mu, H) = \{\mathbf{x}^\nu \mid \mathbf{x}^\nu/\mathbf{x}^\mu \in \mathbb{K}[L(\mathbf{x}^\mu, H)]\}$ intersect only trivially as $\mathbf{x}^\mu$ varies in H.
- Filter axiom: If $H \supseteq H'$, then $L(\mathbf{x}^\mu, H') \supseteq L(\mathbf{x}^\mu, H)$.
- Polynomial set $H \subset \mathcal{I} \trianglelefteq \mathcal{R}$ *L-involutive basis* of ideal $\mathcal{I}$, if

$$\bigsqcup_{\mathbf{x}^\mu \in \operatorname{lt}(H)} \mathcal{C}_L(\mathbf{x}^\mu, \operatorname{lt}(H)) = \operatorname{lt}(\mathcal{I}).$$

Every involutive basis is a Gröbner basis.

Pommaret Bases and Quasi-stable Ideals

Definition

Pommaret division P: term $\mathbf{x}^\mu$ with $\mu = (0, \ldots, 0, \mu_k, \ldots, \mu_n)$ and $\mu_k > 0$ $\rightsquigarrow P(\mathbf{x}^\mu) = \{x_1, \ldots, x_k\}$. Define $\mathrm{cls}(\mathbf{x}^\mu) = \min\{k \mid \mu_k > 0\}$.

Remark

Not every monomial ideal $\mathcal{I}$ has a finite Pommaret basis. A monomial ideal with finite Pommaret basis is called quasi-stable.

Definition

A polynomial ideal $\mathcal{I}$ is in *quasi-stable position*, if $\mathrm{lt}(\mathcal{I})$ is quasi-stable.

Applications

From degrevlex Pommaret basis, can read off projective dimension, Castelnuovo–Mumford regularity, and further homological information for $\mathcal{R}/\mathcal{I}$ (e.g. [Seiler 2009]).

Example: Pommaret Basis

Example

- Consider $\mathcal{I} = \langle x^2, y^2 \rangle \trianglelefteq \mathbb{K}[x, y]$, we have $\mathrm{cls}(x^2) = 1$, $\mathrm{cls}(y^2) = 2$.
- We obtain a Pommaret basis by adding the generator $x^2 y$.

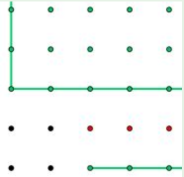

Figure: The ideal $\mathcal{I}$ and the Pommaret cones of its generators

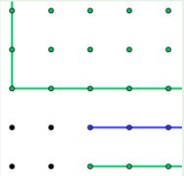

Figure: The Pommaret cone of $x^2 y$ added.

Janet-like Division

The *Janet-like division* was defined by Gerdt and Blinkov in 2005.

- Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$.
- For $d_i, \ldots, d_n \in \mathbb{Z}_{\geq 0}$, define a *Janet class* of U:

$$U_{[d_i,\ldots,d_n]} := \left\{u \in U \mid \deg_j(u) = d_j,\ i \leq j \leq n\right\} \subseteq U\ .$$

- The power $x_i^{k_i}$ with

$$k_i = \min\left\{\deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1},\ldots,d_n]}, \deg_i(v) > \deg_i(u)\right\}$$

  is called a non-multiplicative power of $u \in U_{[d_{i+1},\ldots,d_n]}$ if $k_i > 0$.
- The set $\mathrm{NMP}(u, U)$ collects all non-multiplicative powers of $u \in U$.
- J-multipliers for $u \in U$: terms not divisible by any $x_i^{k_i} \in \mathrm{NMP}(u, U)$.
- $u \in U$ is Janet-like divisor of $w \in \mathcal{T}$, if w/u is a J-multiplier for u.

Janet-like Tree

Given $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$, the Janet-like tree of U contains nodes with labels (t, V, M), where. . .

1. . . . $t \in \mathcal{T}$ encodes a Janet class of U,
2. . . . V is the set of variables multiplicative for each term in this class.

Janet-like Tree

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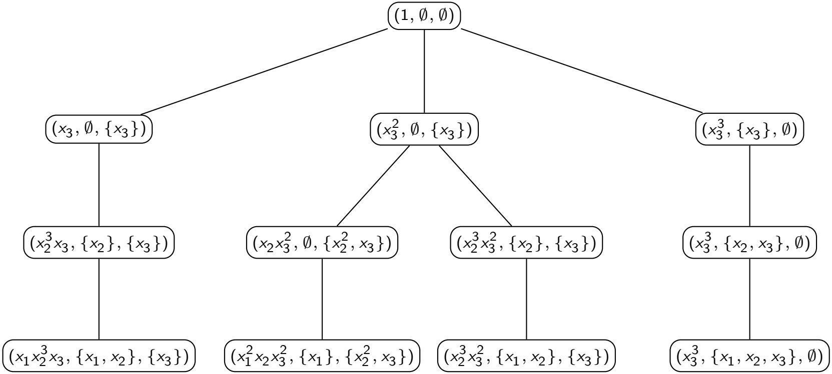
1. ... $t \in \mathcal{T}$ encodes a Janet class of U,
2. ... V is the set of variables multiplicative for each term in this class.
3. ... M is the set of non-multiplicative powers common to all terms in this class.

Janet-like Tree of Janet-like Basis (1)



Janet-like tree of Janet-like basis

$$B = \{x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3].$$

Cones and Complementary Decompositions

- Cone: set of terms $\mathcal{C}_Y(t) = \mathcal{T}_Y \cdot t$, where $t \in \mathcal{T}$ vertex and $\mathcal{T}_Y$ set of terms of $\mathbb{K}[Y]$, with $Y \subseteq \{x_1, \ldots, x_n\}$.
- Monomial ideal $\mathcal{I}$: Involutive bases decompose $\mathcal{I} \cap \mathcal{T}$ into cones.
- Cone decomposition of $\mathcal{T} \setminus \mathcal{I}$: Complementary decomposition (C.D.).

Definition

Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal. The *Hilbert function* of $\mathcal{I}$ is:
$\mathrm{HF}_{\mathcal{I}} : \mathbb{N}_0 \to \mathbb{N}_0,\ d \mapsto |\{t \in \mathcal{T} : \deg(t) = d \wedge t \notin \mathcal{I}\}|$.

For $d \gg 0$, $\mathrm{HF}_{\mathcal{I}}(d) = \mathrm{HP}_{\mathcal{I}}(d)$, with $\mathrm{HP}_{\mathcal{I}} \in \mathbb{Q}[x]$ *Hilbert polynomial* of $\mathcal{I}$.

Application

C. D. of $\mathcal{I}$ easily yields its dimension, its Hilbert function and polynomial.

Compressed Complementary Decomposition

Algorithm 1: Complementary decomposition (from Janet-like basis)

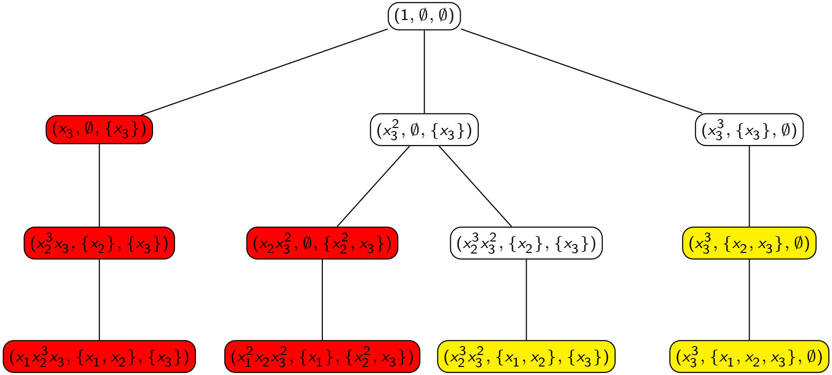
Data: Janet-like tree JLT of Janet-like basis of monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$

Result: Finite complementary decomposition D of $\mathcal{I}$

begin
- $D \longleftarrow \emptyset$
- **for** $k = n, \ldots, 1$ **do**
  - **foreach** *node* $(\mathbf{x}^\rho, V', M')$ *at level* $k+1$ *in JLT* **do**
    - let $(x_k^m \mathbf{x}^\rho, V, M)$ be the leftmost child of $(\mathbf{x}^\rho, V', M')$
    - **if** $m > 0$ **then**
      - $N \longleftarrow \{x_1, \ldots, x_{k-1}\} \cup V'$
      - $D \longleftarrow D \cup \left\{ (\mathbf{x}^\rho \mathbf{x}^\mu, N) : \mathbf{x}^\mu | x_k^{m-1} \prod_{x_a^{h_a} \in M'} x_a^{h_a - 1} \right\}$
- **return** D

(From [Hashemi et al. 2022])

Janet-like Tree of Janet-like Basis (2)



Nodes in the Janet-like tree of $B = \{x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3\}$ that contribute cones to the C.D. of $\mathcal{I} = \langle B \rangle$. (In red.)

Example Complementary Decomposition

Example

Using algorithm and Janet-like tree, obtain following complementary decomposition for $\mathcal{I} = \langle x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3 \rangle$. Cones are given as pairs of vertex and set of multiplicative variables.

$$
\begin{aligned}
&(1, \{x_1, x_2\}),\\
&(x_3, \{x_1\}), (x_2 x_3, \{x_1\}), (x_2^2 x_3, \{x_1\}),\\
&(x_2^3 x_3, \{x_2\}),\\
&(x_3^2, \{x_1\}),\\
&(x_2 x_3^2, \emptyset), (x_2^2 x_3^2, \emptyset), (x_1 x_2 x_3^2, \emptyset), (x_1 x_2^2 x_3^2, \emptyset).
\end{aligned}
$$

The cones in each line are encoded in a single node of the Janet-like tree.

Involutive-like Divisions and Pommaret-like Division

Definition

Involutive-like division L on $\mathcal{T}$, given finite set $U \subset \mathcal{T}$ and $u \in U$, assigns:

- the *non-multiplicative powers* $\mathrm{NMP}_L(u, U)$ of $u \in U$
- the set of *L-non-multipliers* $\bar{L}(u, U) = \langle \mathrm{NMP}_L(u, U) \rangle$
- The set of *L-multipliers* $L(u, U)$, $\mathcal{T} \setminus \bar{L}(u, U)$.
- For any $u \in U$, the *involutive cone* $\mathcal{C}_L(u, U) = u \cdot L(u, U)$.

The cones must intersect only trivially. No filter axiom assumed.

Definition

The *Pommaret-like division* P assigns $\mathrm{NMP}_P(u, U)$ as follows:

- All Janet-like non-multiplicative powers $x_a^{p_a}$ with $a > \mathrm{cls}(u)$.
- The Janet multiplicative variables x_b with $b > \mathrm{cls}(u)$.

Pommaret-like Bases

Theorem (Hashemi et al. 2023)

1. *A monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is quasi-stable, if and only if it possesses a finite Pommaret-like basis (PLB).*
2. *From monomial PLB U of $\mathcal{I}$, obtain Pommaret basis U' of $\mathcal{I}$:*

$$U' = \left\{ t \cdot \mathbf{x}^{\mu} : t \in U \wedge \mathbf{x}^{\mu} \mid \prod_{x_a^{p_a} \in \mathrm{NMP}_P(t,U)} x_a^{p_a - 1} \right\}.$$

3. *The minimal Janet-like basis of a quasi-stable monomial ideal $\mathcal{I}$ is also a PLB of $\mathcal{I}$.*

Application

Pommaret-like bases...

- enjoy compactness and combinatorial machinery of Janet-like bases,
- contain all the information a Pommaret basis contains.

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Monomial Curves

A monomial curve in the projective plane $\mathbb{P}^2_{\mathbb{K}}$ is an algebraic curve parameterized by a map

$$p : \mathbb{P}^1_{\mathbb{K}} \longrightarrow \mathbb{P}^2_{\mathbb{K}}, (T_0 : T_1) \mapsto (T_0^d : T_0^u T_1^{d-u} : T_1^d),$$

where $d \in \mathbb{Z}_{>1}$ is the degree of the curve.
Associated equigenerated monomial ideal $I = \langle T_0^d, T_0^u T_1^{d-u}, T_1^d \rangle$.

- Similarly, can define monomial curves in spaces of higher dimensions.
- The theory encompasses well-studied special cases: For example, the rational normal space curve is the twisted cubic.

Rees Ideals and Algebras

The Rees algebra associated to I is $\mathcal{R}_I = \oplus_{m \geq 0} I^m \cdot z^m \subset \mathbb{K}[T_0, T_1, z]$, where z is a new variable. The Rees ideal $\mathcal{R}(I)$ is the kernel of the ring homomorphism $\phi : \mathbb{K}[T_0, T_1, X_0, X_1, X_2] \to \mathbb{K}[T_0, T_1, z]$, defined by:

$$\begin{aligned} X_0 &\mapsto T_0^d z, \\ X_1 &\mapsto T_0^u T_1^{d-u} z, \\ X_2 &\mapsto T_1^d z. \end{aligned}$$

The structure of the Rees algebra is best known for plane curves (e.g. [Cortadellas Benítez, D'Andrea 2015]).

Minimal Generators of $\mathcal{R}(I)$

- $q = \frac{d}{\gcd(d,u)}$ and $[q] = \{1, \ldots, q\}$.
- $a_i = u \cdot i \pmod d$ and $b_i = \lfloor \frac{u.i}{d} \rfloor$ for $i = 1, 2, \ldots, q$.
- $\Delta_{\max}(d,u) = \{i \in [q] \mid a_i = \max\{a_1, \ldots, a_i\}\}$ and $\Delta_{\min}(d,u) = \{i \in [q] \mid a_i = \min\{a_1, \ldots, a_i\}\}$

Lemma (Iglesias et al. 2024)

1. *$Syz(I^j)$ with $j \in \Delta_{\max}$ induces the minimal generator*

$$T_1^{d-a_j} \cdot X_0^{b_j+1} \cdot X_2^{j-(b_j+1)} - T_0^{d-a_j} \cdot X_1^j.$$

2. *$Syz(I^j)$ with $j \in \Delta_{\min}$ induces the minimal generator*

$$T_1^{a_j} \cdot X_1^j - T_0^{a_j} \cdot X_0^{b_j} \cdot X_2^{j-b_j}.$$

3. *There are no other minimal generators.*

Example

Let $I = \langle T_0^{21}, T_0^{15}T_1^6, T_1^{21}\rangle$, so $d = 21$ and $u = 15$. From $\mathrm{Syz}(I)$, we add:

$$g_1 = T_1^6 X_0 - T_0^6 X_1 \text{ and } g_2 = T_1^{15} X_1 - T_0^{15} X_2$$

Now consider I^t where $t > 1$.
We have $a_1 = 15$; for $t = 2$, $a_2 = 9 = \min\{a_1, a_2\}$, yielding for $\mathrm{Syz}(I^2)$:

$$g_3 = T_1^9 X_1^2 - T_0^9 X_0 X_2$$

For $t = 3$, $a_3 = 3 = \min\{a_1, \ldots, a_3\}$, so we get:

$$g_4 = T_1^3 X_1^3 - T_0^3 X_0^2 X_2^1$$

For $t = 4$, $a_4 = 18 = \max\{a_1, \ldots, a_4\}$, so we get:

$$g_5 = T_1^3 X_0^3 X_2^1 - T_0^3 X_1^4$$

For $t = 5$ and $t = 6$, $a_5 = 12$ and $a_6 = 6$, being neither minimal nor maximal values. The last relevant syzygy is from $\mathrm{Syz}(I^7)$:

$$g_6 = X_1^7 - X_0^5 X_2^2$$

Associated Graph

Proposition (Iglesias et al. 2024)

Any $\delta \in \Delta_{\mathrm{min}} \cup \Delta_{\mathrm{max}}$ *with* $\delta > 1$ *is given as* $\delta = \beta + \gamma$, *where:*

$$\beta = \max\{\alpha \in \Delta_{\mathrm{min}} \mid \alpha < \delta\},$$
$$\gamma = \max\{\alpha \in \Delta_{\mathrm{max}} \mid \alpha < \delta\}$$

We can then divide the minimal binomial generators into two types (*lower* or *upper*) and build graphs of the following form:

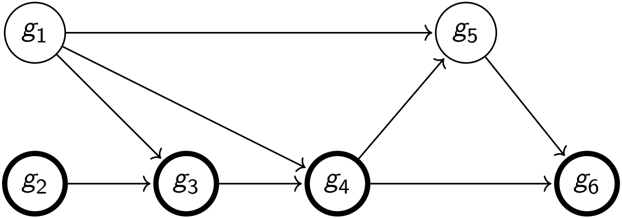

Figure: $\mathrm{Graph}(\mathcal{R}(I))$ for parameters $d = 21$, $u = 15$.

Free Resolution from Graph

Write $P = \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$.
Let r be the number of arrows in the directed graph associated to the minimal generating set G of $\mathcal{R}(I)$ and s the number of triangles. The minimal P-free resolution of $\mathcal{R}(I)$ is of the form

$$0 \to P^s \xrightarrow{\Phi_2} P^r \xrightarrow{\Phi_1} P^{|G|} \xrightarrow{\Phi_0} \ker(\phi) \to 0.$$

- The resolution contains exactly one first syzygy for each arrow.
- The resolution contains exactly one second syzygy for each triangle.
- There is an explicit formula for the differential maps Φ_i which can be read off from the graph.
- The first syzygies have exactly 3 nonzero terms and the second syzygies have exactly 4 nonzero terms.

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The Janet-like Basis

- We use the degree reverse lexicographic term ordering $\prec$ with $X_2 \prec X_0 \prec T_0 \prec T_1 \prec X_1$
- $\prec$ is optimal w.r.t. the goal of obtaining a Pommaret basis.
- For getting a Gröbner basis $\overline{G}$ for $\mathcal{R}(I)$, we only add one element representing the trivial syzygy in $Syz(I^1)$ between T_0^d and T_1^d to the minimal generating set G.
- The corresponding binomial is $g_0 = T_1^d \cdot X_0 - T_0^d \cdot X_2$.

Proposition (Iglesias et al. 2024)

Let $\overline{G} = \{g_0, g_1, \ldots, g_r\}$ be the Gröbner basis of $\mathcal{R}(I)$. Then, $H := \overline{G} \cup \{\hat{g}_3, \ldots, \hat{g}_{r-1}\}$ is the minimal Janet-like basis w.r.t. $\prec$, where: $\hat{g}_\ell = g_m * X_1^{i-j}$ for every $\ell \in \{3, \ldots, r-1\}$ such that g_m is of the opposite type w.r.t. g_ℓ (lower/upper), and the value of j with $j := \deg_{X_1}(g_m) < \deg_{X_1}(g_\ell) =: i$ is the maximal possible.

Example

We continue our running example. Again consider $I = \langle T_0^{21}, T_0^{15}T_1^6, T_1^{21}\rangle$ and the minimal generating set of $\mathcal{R}(I)$, $G = \{g_1, g_2, g_3, g_4, g_5, g_6\}$. We get the following minimal Janet-like basis w.r.t. $\prec$:

$$H = \{g_0, g_1, g_2, \hat{g}_3, g_3, \hat{g}_4, g_4, g_5, \hat{g}_5, g_6\},$$

where

$g_0 = T_1^{21}X_0 - T_0^{21}X_2$, $\hat{g}_3 = g_1 * X_1 = T_0^6X_1^2 - T_1^6X_0X_1$,
$\hat{g}_4 = g_1 * X_1^2 = T_0^6X_1^3 - T_1^6X_0X_1^2$, $\hat{g}_5 = g_4 * X_1 = T_1^3X_1^4 - T_0^3X_0^2X_1X_2$.

The non-multiplicative powers (w.r.t. $\mathrm{lt}(H)$) for every element in $\mathrm{lt}(H)$ are:

$\mathrm{NMP}(g_0) = \{X_1\}$, $\mathrm{NMP}(g_1) = \{X_1, T_1^{15}\}$, $\mathrm{NMP}(g_2) = \{X_1\}$,
$\mathrm{NMP}(\hat{g}_3) = \{X_1, T_1^9\}$, $\mathrm{NMP}(g_3) = \{X_1\}$, $\mathrm{NMP}(\hat{g}_4) = \{X_1, T_1^3\}$,
$\mathrm{NMP}(g_4) = \{X_1\}$, $\mathrm{NMP}(g_5) = \{X_1^3, T_1^3\}$, $\mathrm{NMP}(\hat{g}_5) = \{X_1^3\}$,
$\mathrm{NMP}(g_6) = \emptyset$.

Extended Associated Graph

- Like for the minimal generators, we associate a directed graph to the Janet-like basis H.
- In this graph, arrows correspond to non-multiplicative powers.
- The graph's overall structure is the same for all parameter values d, u.

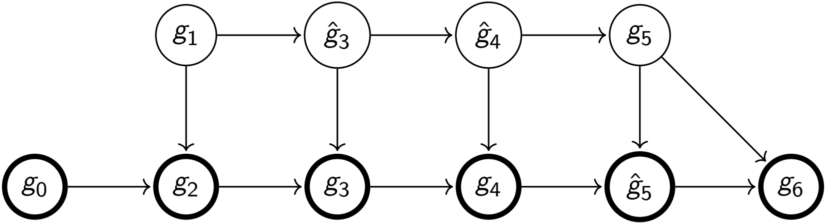

Figure: Directed graph associated to the Janet-like basis of $\mathcal{R}(I)$ for parameteres $d = 21$, $u = 15$. The arrows are in bijection to the set of all NMPs.

Invariance of Janet-like Tree Structure

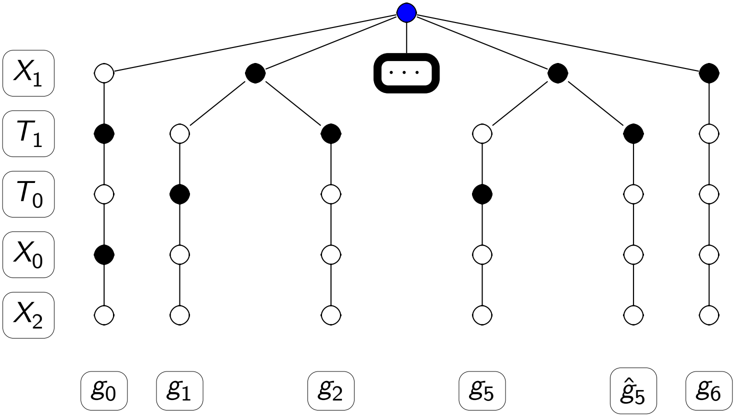

Figure: Shape of the Janet-like tree for $d = 21$, $u = 15$. Not drawn: The branches corresponding to $g_3, \hat{g}_3, g_4, \hat{g}_4$. All plane parametrization ideals have trees of the same shape, with repetitions of the inner blocks.

Complementary Decomposition and Application

For any plane parameterization ideal I, the Janet-like tree yields for the Rees ideal $\mathcal{R}(I)$:

- In the complementary decomposition of $\mathrm{lt}(\mathcal{R}(I))$, there is only one group of cones with three free variables, all other cones have less.
- $\dim(\mathcal{R}(I)) = 3$.
- g_0 is the only obstruction to quasi-stable position.
- In quasi-stable position, need pure power of T_1 in a leading term.

Theorem (Iglesias et al. 2024)

The linear change $X_0 \to X_0 + T_i$ and $X_2 \to X_2 + T_i$ gives coordinates for which $\mathcal{R}(I)$ is in quasi-stable position for any $i \in \{0, 1\}$, w.r.t. the degrevlex ordering with $X_2 \prec X_0 \prec T_{1-i} \prec T_i \prec X_1$.

Projective Dimension and Regularity

Theorem (Iglesias et al. 2024)

Consider the parameterization ideal I, with parameters d and u, of a plane monomial curve and its Rees ideal $\mathcal{R}(I)$.

1. *If* $d \neq 2u$, *then* $\mathrm{projdim}(\mathcal{R}(I)) = 3$. *Otherwise,* $\mathrm{projdim}(\mathcal{R}(I)) = 2$.
2. *If* $d \neq 2u$, *then:*
   - *if* $\gcd(d, u) = 1$, *then* $\mathrm{reg}(\mathcal{R}(I)) = d$;
   - *if* $\gcd(d, u) > 1$, *then* $\mathrm{reg}(\mathcal{R}(I)) = \max\{u, d - u\} + 1$.

   If $d = 2u$, *then* $\mathrm{reg}(\mathcal{R}(I)) = u + 1$.

Corollary

$\mathcal{R}(I)$ is Cohen–Macaulay if and only if $d = 2u$.

Curves in Higher-dimensional Space

The following is supported by example computations using the CAS `Macaulay2` and its package `InvolutiveBases`:

Conjecture

Let I be the monomial ideal associated to a monomial curve in a projective space of dimension $n \geq 2$. Then:

1. $\dim \mathcal{R}(I) = 3$.
2. The linear change $X_0 \to X_0 + T_i$ and $X_n \to X_n + T_i$ gives coordinates for which $\mathcal{R}(I)$ is in quasi-stable position for any $i \in \{0, 1\}$, w.r.t. the degrevlex ordering with $X_n \prec X_0 \prec T_{1-i} \prec T_i \prec X_1 \prec \cdots \prec X_{n-1}$.
3. The maximal degrees D in the minimal generating set and the Pommaret-like basis of $\mathcal{R}(I)$ coincide, whence $D = \mathrm{reg}(\mathcal{R}(I))$.

Conclusion

- Advantage Obtaining Pommaret-like bases for Rees ideals of monomial curves yields a method for studying their homological properties which is independent of the space dimension.
- Drawback Lose binomiality by applying linear coordinate change, even if very sparse.

Thank you for your attention!